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Random hierarchical matrices: spectral properties and relation to polymers on disordered trees

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Abstract

We study the statistical and dynamic properties of the systems characterized by an ultrametric space of states and translationary non-invariant symmetric transition matrices of the Parisi type subjected to 'locally constant' randomization. Using the explicit expression for eigenvalues of such matrices, we compute the spectral density for the Gaussian distribution of matrix elements. We also compute the averaged 'survival probability' (SP) having sense of the probability of finding a system in the initial state by time t. Using the similarity between the averaged SP for locally constant randomized Parisi matrices and the partition function of directed polymers on disordered trees, we show that for times $t > t_{cr}$ (where t_{cr} is some critical time) a 'lacunary' structure of the ultrametric space occurs with the probability $1 - \text{const}/t$. This means that the escape from some bounded areas of the ultrametric space of states is locked and the kinetics is confined in these areas for an infinitely long time.

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The explosion of the interest in the statistical properties of ensembles of random matrices around the 1950s was motivated, in the first turn, by a series of physical problems in nuclear physics. Later, physical applications of the random matrix theory (RMT) were extended to mesoscopics and statistical physics. One can mention recent works [1, 2], which deal with physical applications of the eigenvalue statistics of ensembles of random matrices. The typical problems of RMT concern the evaluation of the averaged spectral density, as well as distributions of 'level spacings' under the supposition that all matrix elements are independent randomly distributed entries which take values in some specific ensemble [3].

Despite the fact that the standard RMT can be applied to a wide range of physical phenomena, RMT, in its basic form, does not cover a particular class of *hierarchical* complex systems important for glasses, real networks and proteins. The description of such complex

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systems deals with the concept of hierarchical organization of energy landscapes [4, 5]. A complex system is assumed to have a large number of metastable states corresponding to local minima in the potential energy landscape. With respect to the transition rates, the minima are suggested to be clustered in hierarchically nested basins of minima, i.e. larger basins consist of smaller basins, each of these consists of even smaller ones and so on. The basins of local energy minima are separated by a hierarchically arranged set of barriers: large basins are separated by high barriers and smaller basins within each larger one are separated by lower barriers. Since the transitions between the basins are determined by the passages over the highest barriers separating them, the transitions between any two local minima obey the 'strong triangle inequality' for *ultrametric distances*. Hence, the hierarchy of basins possesses ultrametric geometry which has natural visualization by the uniform p-adic (i.e. $p+1$-branching) Cayley tree.

It might be said that the systems possessing the ultrametric geometry are hardly compatible with the systems possessing the Euclidean geometry. Recently, however, the interrelation between hyperbolic (i.e. tree-like) and Euclidean geometries has become a topic of physics of disordered systems. It has been shown in [6] that one can construct a random Gaussian translation-invariant potential landscape which reproduces all the essential features of the Parisi landscapes directly in any finite–dimensional Euclidean space. Also, the question of isometric embedding of a uniform Cayley tree into the 3D Euclidean space has been discussed in the work [7]. The concept of ultrametricity has been implemented in a number of toy models describing transport phenomena and the so-called basin-to-basin kinetics of disordered complex systems [8–10]. These models deal with various aspects of *ultrametric diffusion*—a certain type of stochastic motion in a *nonrandom* energy landscape with regular hierarchy encoded in the symmetric Parisi transition matrix, $\bar{T}$. The model of ultrametric diffusion has been successfully applied to the dynamics of proteins [11]. The eigenvalues, $\{\bar{\lambda}_j\}$, of the matrix $\bar{T}$ determine a hierarchy of relaxation times of the entire system and, hence, define the kinetics constrained to an ultrametric landscape. The values of $\bar{\lambda}_j$ are well known [8, 11, 13]:

$$\bar{\lambda}_j = -p^{\gamma}\bar{T}^{(\gamma)} - (1-p^{-1})\sum_{\gamma'=\gamma+1}^{\Gamma} p^{\gamma'}\bar{T}^{(\gamma')}, \tag{1}$$

where $\bar{\lambda}_0 = 0$ by definition, $\bar{T}^{(\gamma')}$ is the element of the Parisi matrix $\bar{T}$ related to the hierarchical level γ of the p-adic tree and the summation runs up to the maximal level $\max[\gamma] = \Gamma$. Since $\bar{T}$ is a kinetic matrix, the sum of all matrix elements in each column of $\bar{T}$ is equal to zero.

Structural randomness is inherent for any disordered system. Thus, the representation of the energy landscape by a *regular* hierarchy is rather ambiguous. In some cases, like in spin–glass models, one supposes that the distribution of energy barriers is fully random. However, for ultrametric energy landscapes it is more reasonable to suppose that the barriers, belonging to the same hierarchical level, coincide only on the order of magnitude. So, it is natural to think that the transition rates between basins of the *same* hierarchical level have random distribution around definite mean values prescribed by the ultrametric hierarchy. Besides that, in real experiments we usually do not deal with an individual molecule, but with an ensemble of molecules in the sample. Therefore, the computation of an observable is connected with the averaging of a dynamics in an individual ultrametric landscape over the ensemble of its realizations in the sample.

To specify the model, consider two basins, belonging to the same hierarchal level. The dynamic trajectories leading from one basin to the other pass through the common 'saddle point'. The transition rates between the states in these basins depend only on the location of the 'saddle point', but not on the details of the behavior of the system in the basins. This

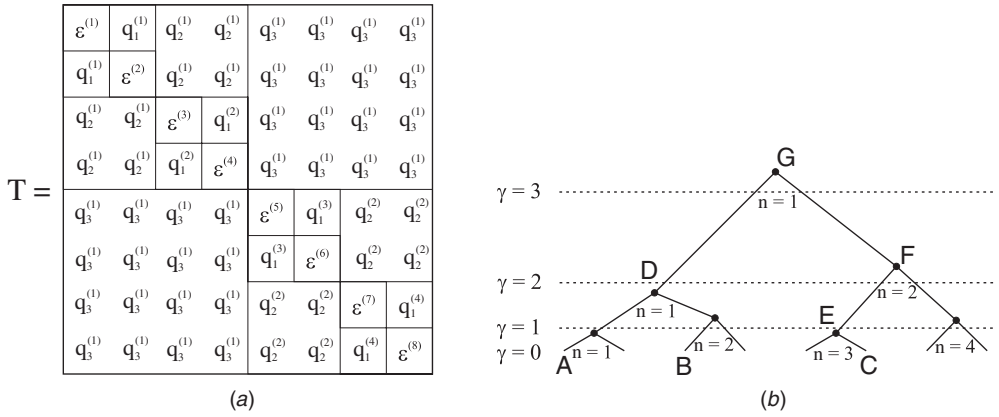

Figure 1. (*a*) Locally constant randomized Parisi matrix ($p = 2$) and (*b*) locally constant landscape corresponding to the LCP matrix ($p = 2$).

scenario implies that all elements of the Parisi-type matrix are always equal inside a block, but they can fluctuate from one block to another. Thus, we arrive at the randomized 'locally constant Parisi (LCP) matrix', T, which is depicted in figure 1(*a*). Recall that T is the kinetic matrix; hence, the sum of matrix elements in each column is zero.

Each matrix element, $T_{i,j} \equiv q_\gamma^{(n)}$, defines the probability of passing over the corresponding barrier. Hence, the dimensionless barrier energy (the 'barrier height'), $u_\gamma^{(n)} = -\ln q_\gamma^{(n)}$ $\left(u_\gamma^{(n)} > 0\right)$, can be introduced. As has been pointed above, we suppose that

$$u_\gamma^{(n)} = \left\langle u_\gamma^{(n)} \right\rangle + \xi_\gamma^{(n)}, \tag{2}$$

where $\left\langle u_\gamma^{(n)} \right\rangle$ is the mean barrier height in a given hierarchy level, γ, and $\xi_\gamma^{(n)}$ describes the fluctuations around this mean value. The mean values $\left\langle u_\gamma^{(n)} \right\rangle$ fix the structure of the nonrandom (i.e. regular) Parisi matrix. If the barrier heights grow (in average) linearly with γ, we set $\mathrm{e}^{-\left\langle u_\gamma^{(n)} \right\rangle} = \left\langle q_\gamma^{(n)} \right\rangle = p^{-(\alpha+1)\gamma}$, where α is a linear landscape 'slope' ($\alpha \geqslant 0$) [8, 11].

Thus, the problem of stochastic motion in the randomized ultrametric landscapes has appeared. The first important question concerns the computation of the spectral density of the system described by the randomized Parisi-type symmetric matrix T shown in figure 1(*a*). Besides that, we analyze some probabilistic properties of the averaged 'survival probability' using the analogy with the statistics of polymers on disordered trees. For any distribution of $\xi_\gamma^{(n)}$ the spectral density of the matrix T can be, in principle, computed numerically. However, since the locally constant Parisi matrix T defines the new class of random matrices not yet considered in the literature, it is worth understanding the analytic structure of spectral density of T for the Gaussian distribution of $\xi_\gamma^{(n)}$. In what follows, we suppose that $\left|\xi_\gamma^{(n)}\right| \ll 1$ in (2). This allows us to truncate the power series for $q_\gamma^{(n)}$ in $\xi_\gamma^{(n)}$ at the linear term:

$$q_\gamma^{(n)} \approx \mathrm{e}^{-\left\langle u_\gamma^{(n)} \right\rangle}\left(1 - \xi_\gamma^{(n)} + O\left(\left(\xi_\gamma^{(n)}\right)^2\right)\right). \tag{3}$$

The non-negativity of the matrix elements of T is ensured by the condition $\left|\xi_\gamma^{(n)}\right| \ll 1$.

The randomized matrix T can be viewed as an ultrametrically organized system of barriers defined by a nonuniform p-adic tree—see figure 1(*a*) ($p = 2$). Each matrix element of T represents a transition probability over the largest barrier separating any two different states located on the lowest level $\gamma = 0$. For example, the states A and B are separated by the barrier

in the branching point D and the transition probability from A to B is $q_2^{(1)}$; the states A and C are separated by the barrier in the branching point G and the transition probability from A to C is $q_3^{(1)}$ (see figure 1(*a*)).

The eigenvalues of the LCP matrix have been derived for the first time in [12] using the elements of the *p*-adic analysis [13]. The construction of $\lambda_{\gamma,n}$ has a very transparent geometric interpretation. Define a pair of numbers (γ, n), where γ is the hierarchy level $(1 \leqslant \gamma \leqslant \Gamma = \max[\gamma])$ and n enumerates the blocks in the same hierarchy level $(1 \leqslant n \leqslant p^{\Gamma-\gamma})$, as is shown in figure 1(*a*). Then, the eigenvalue $\lambda_{\gamma,n}$ of the LCP matrix for the *p*-adic tree reads as

$$\lambda_{\gamma,n} = -p^{\gamma} q_{\gamma}^{(n)} - (1 - p^{-1}) \underbrace{\sum_{\gamma'=\gamma+1}^{\Gamma} p^{\gamma'} q_{\gamma}^{(n')}}_{\Sigma}. \tag{4}$$

In equation (4) $q_{\gamma}^{(n)}$ are the elements of the matrix T shown in figure 1(*a*), while the summation Σ (despite that it resembles (1)) contains some ambiguities. It is clear from figure 1(*a*) that any pair (γ, n) fixes some vertex of the non-regular *p*-adic tree. The sum Σ in (4), contributing to $\lambda_{\gamma,n}$, runs from the point (γ', n'), which is the upper nearest neighbor of the point (γ, n) along a *p*-adic tree, towards the root point $(\Gamma, 1)$. For example, let us compute the eigenvalue $\lambda_{\gamma=1,n=3}$, $p = 2$ (see figure 1(*b*)). The first term in (4) is just the weighted contribution from the point $E(\gamma = 1, n = 3)$, while the sum Σ in this case is the weighted sum of two terms coming from the points F and G (see figure 1(*a*)). Hence, $\lambda_{1,3} = -2^1 q_1^{(3)} - (1 - 2^{-1})\left[2^2 q_2^{(2)} + 2^3 q_3^{(1)}\right]$.

Let $P\left(\xi_{\gamma}^{(n)}\right)$ be the distribution function of the fluctuations $\xi_{\gamma}^{(n)}$ of matrix elements $q_{\gamma}^{(n)}$ (see (3)). The spectral density, $\rho(\lambda)$, of the matrix T can be computed in the standard way:

$$\rho(\lambda) = \frac{1}{\mathcal{N}} \sum_{\{\gamma,n\}} \langle \delta(\lambda - \lambda_{\gamma,n}) \rangle, \tag{5}$$

where $\mathcal{N} = p^{\Gamma}$ is the total number of eigenvalues of the matrix T and $\langle \cdots \rangle$ means averaging with the distribution function $P\left(\xi_{\gamma}^{(n)}\right)$.

Suppose that the distribution of $\xi_{\gamma}^{(n)}$ is Gaussian and does not depend either on the hierarchy level, γ, or on the index n counting different blocks in the same hierarchy level:

$$P\left(\xi_{\gamma}^{(n)}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(\xi_{\gamma}^{(n)}\right)^2}{2\sigma^2}\right) s \tag{6}$$

Speaking formally, since the fluctuations of $\xi_{\gamma}^{(n)}$ in (6) are not restricted, the distribution $P\left(\xi_{\gamma}^{(n)}\right)$ in equation (6) is inconsistent with the condition $\left|\xi_{\gamma}^{(n)}\right| \ll 1$ introduced above. Recall that the last condition ensures the matrix elements $q_{\gamma}^{(n)}$ in (3) to be non-negative. However in what follows we suppose $\left\langle\left(\xi_{\gamma}^{(n)}\right)^2\right\rangle = \sigma^2 \ll 1$, derive the spectral density and check our results for self-consistency by comparing analytic and numeric results. With (4)–(6) in hand, we can easily compute the spectral density, $\rho(\lambda)$. First, proceed with an auxiliary computation of $Q(\lambda, \gamma) = \langle \delta(\lambda - \lambda_{\gamma,n=1}) \rangle$ for $n = 1$ and arbitrary γ $(1 \leqslant \gamma \leqslant \Gamma)$, where the averaging is carried out with $P\left(\xi_{\gamma}^{(n=1)}\right)$. The straightforward computations give the following result for $Q(\lambda, \gamma)$ at $\sigma \ll 1$:

$$\begin{aligned} Q(\lambda, \gamma) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi} \mathrm{e}^{\mathrm{i}\lambda x} \langle \mathrm{e}^{-\mathrm{i}\lambda_{\gamma,n} x} \rangle_{\{P(\xi_{\gamma}^{(n=1)}),\ldots,P(\xi_{\Gamma}^{(n=1)})\}} \\ &\approx \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi} \exp\left\{\mathrm{i}x[\lambda + v_{\gamma}(\Gamma)] - \frac{x^2}{2}\sigma^2 u_{\gamma}(\Gamma)\right\}, \end{aligned} \tag{7}$$

where

$$v_\gamma(\Gamma) = p^{-\alpha\gamma} + (1-p^{-1}) \sum_{\gamma'=\gamma+1}^{\Gamma} p^{-\alpha\gamma'},$$
$$u_\gamma(\Gamma) = p^{-2\alpha\gamma} + (1-p^{-1})^2 \sum_{\gamma'=\gamma+1}^{\Gamma} p^{-2\alpha\gamma'}. \tag{8}$$

The integral in (7) can be easily evaluated. Keeping only the leading (quadratic) terms of expansion in σ, we get

$$Q(\lambda,\gamma) = \frac{1}{\sqrt{2\pi\sigma^2 u_\gamma(\Gamma)}} \exp\left\{-\frac{[\lambda + v_\gamma(\Gamma)]^2}{\sigma^2 u_\gamma(\Gamma)}\right\}. \tag{9}$$

Approximation (7) obtained for expansion (3) is valid for $\left\langle \left(\xi_\gamma^{(n)}\right)^2 \right\rangle \ll 1$, i.e. for $\sigma^2 \ll 1$. Let us note that it would be interesting to continue the expansions in (3) and (7) keeping explicitly the terms of order of σ^4. The computation of the spectral density in this case involves the Airy functions. The discussion of this question will be published in the extended version of our work.

Since the distribution function $P\left(\xi_\gamma^{(n)}\right)$ does not depend on n, expressions (7)–(9) hold for any of $1 \leqslant n \leqslant p^{\Gamma-\gamma}$ eigenvalues on the hierarchical level γ. Introducing the number of eigenvalues on the level γ (i.e. the degeneracy), $g(\gamma) = p^{\Gamma-\gamma}$, we can now rewrite the spectral density, $\rho(\lambda)$, as follows:

$$\rho(\lambda) = \frac{1}{p^\Gamma} \sum_{\gamma=1}^{\Gamma} g(\gamma) Q(\lambda,\gamma). \tag{10}$$

We have computed numerically $\rho(\lambda)$ for the Gaussian distribution $P\left(\xi_\gamma^{(n)}\right)$, setting explicitly $q_\gamma^{(n)} = p^{-(\alpha+1)\gamma} \mathrm{e}^{-\xi_\gamma^{(n)}}$. Simultaneously, we have considered the linearized case of matrix elements, $q_\gamma^{(n)} \approx p^{-(\alpha+1)\gamma}\left(1-\xi_\gamma^{(n)}\right)$. We have compared in figure 2(*a*) the explicit and linearized cases for $\Gamma = 8$ and $\alpha = 0.1$. The same results for $\alpha = 0.5$ are shown in figure 2(*b*). We see that for $\sigma \lesssim 0.2$, the linearization of the matrix elements suggested in (3) still makes sense for the Gaussian distribution of energies. The analytic results computed using (9) and (10) coincide with the linearized case and are indistinguishable in the figures.

The analysis of (10) for arbitrary $\alpha > 0$ in the limit $\Gamma \to \infty$ allows one to extract the asymptotic behavior of the tail of the spectral density $\rho(\lambda)$ as $\lambda \to -\infty$:

$$\rho(\lambda)|_{\lambda\to-\infty} \approx \begin{cases} |\lambda|^{-\frac{\alpha-1}{\alpha}} \mathrm{e}^{-\lambda^2} & \text{for} \quad \alpha \geqslant 1 \\ |\lambda|^{-\frac{1-\alpha}{\alpha}} \mathrm{e}^{-\lambda^2} & \text{for} \quad 0 < \alpha < 1. \end{cases} \tag{11}$$

Information about the relaxation characteristics of any system (including the hierarchical one) is encoded into the so-called survival probability, $W(t)$, having a sense of the probability of finding a dynamical system in the initial state by the time t. For a nonrandom kinetic Parisi matrix of Γ hierarchy levels, the survival probability simply is

$$\bar{W}(t,\Gamma) = (p-1) \sum_{\gamma=1}^{\Gamma} p^{-\gamma}\, \mathrm{e}^{\lambda_\gamma t} + p^{-\Gamma},$$

where the eigenvalues λ_γ are defined in (1) (recall that $\lambda_\gamma < 0$ for $\gamma = 1, \ldots, \Gamma$). For LCP transition matrices randomly distributed over the ensemble, the survival probability is a

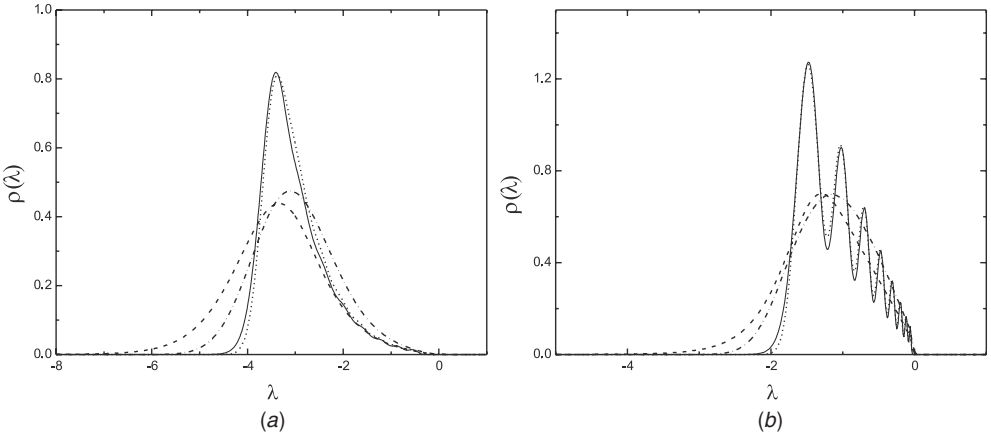

Figure 2. Spectral density $\rho(\lambda)$ for $\Gamma = 8$: (*a*) $\alpha = 0.1$: solid line—explicit case and dotted line—linearized case for $\sigma = 0.2$; dashed line—explicit case and dotted–dashed line—linearized case for $\sigma = 0.5$; (*b*) $\alpha = 0.5$: notations are the same as in figure 2(*a*).

(This figure is in colour only in the electronic version)

random quantity. To make the survival probability independent of the specific initial point on the tree, we consider $W(t, \Gamma)$ averaged over all positions of initial states:

$$\overline{W}(t, \Gamma) = (p-1) \sum_{\{\gamma, n\}} p^{-\gamma} \mathrm{e}^{\lambda_{\gamma,n} t} + p^{-\Gamma} = (p-1) \sum_{\gamma=1}^{\Gamma} p^{-\gamma} Z(t, \gamma, \Gamma) + p^{-\Gamma}, \tag{12}$$

where

$$Z(t, \gamma, \Gamma) = \sum_{n=1}^{p^{\Gamma-\gamma}} \mathrm{e}^{\lambda_{\gamma,n} t}. \tag{13}$$

The sum in (13) runs over all $p^{\Gamma-\gamma}$ tree vertices of the level γ and the eigenvalues, $\lambda_{\gamma,n}$, are defined in (4). Note that despite that $\overline{W}(t, \Gamma)$ does not depend on particular initial states on the tree, it is still a random quantity since $Z(t, \gamma, \Gamma)$ and, hence, $\overline{W}(t, \Gamma)$ depends (via equation (4)) on the ensemble of random barriers, $\left\{\xi_\gamma^{(n)}\right\}$.

The direct computation of the survival probability is a difficult problem, though some important information about the probabilistic behavior of $\langle \overline{W}(t, \Gamma) \rangle$ is available by analyzing the distribution functions of the sub-sums $Z(t, \gamma, \Gamma)$. Note that the eigenvalue, $\lambda_{\gamma,n}$, of the kinetic LCP matrix T defines the escape rate from the 'basin' (i.e. sub-tree) with the root located in the point (γ, n)—see figure 1(*b*). Hence, $Z(t, \gamma, \Gamma)$ is the partition function of all escape rates at time t from the hierarchy level γ.

Further consideration is based on the observation that $Z(t, \gamma, \Gamma)$ resembles the partition function of the directed polymer on the disordered tree (DPDT) analyzed in [14]. Substituting expansion (3) into (13), and keeping only the linear terms in $\xi_\gamma^{(n)}$ $\left(\left|\xi_\gamma^{(n)}\right| \ll 1\right)$, we get $Z(t, \gamma, \Gamma) = \tilde{Z}(t, \gamma, \Gamma)\, \mathrm{e}^{-v_\gamma(\Gamma) t}$, where $v_\gamma(\Gamma)$ is defined in (8) and is independent of any particular path on the tree, while $\tilde{Z}(t, \gamma, \Gamma)$ has the following expression:

$$\tilde{Z}(t, \gamma, \Gamma) = \sum_{\text{all paths}} \exp\left\{ t \left(p^{-\alpha\gamma} \xi_\gamma^{(n)} + (1 - p^{-1}) \sum_{\gamma'=\gamma+1}^{\Gamma} p^{-\alpha\gamma'} \xi_{\gamma'}(n') \right) \right\}. \tag{14}$$

The first summation in (14) is taken over all $p^{\Gamma-\gamma}$ vertices of the γ-level on the p-adic tree, and the sum in the exponent runs along the path on the tree from the vertex (γ, n) towards the root point (exactly as in (4)).

The function $\tilde{Z}(t, \gamma, \Gamma)$ satisfies the stochastic recursion (compare to [14]):

$$\tilde{Z}(\Gamma) = \exp\left[t(1-p^{-1})p^{-\nu\Gamma}\xi_\Gamma\right]\sum_{j=1}^{p}\tilde{Z}_j(\Gamma-1). \tag{15}$$

Introduce as in [14] the averaged characteristic function, $G_m(x)$, by

$$G_m(x) = \langle\exp\{-\tilde{Z}(m)\exp[t(1-p^{-1})p^{-\alpha m}x]\}\rangle, \tag{16}$$

where $\gamma \leqslant m \leqslant \Gamma$. Using the factorization of $G_m(x)$ on the Cayley tree, we come to the recursion for $G_m(x)$:

$$G_{m-1}(x) = \int \mathrm{d}\xi\, P(\xi)[G_m(p^{-\alpha}(x+\xi))]^p, \tag{17}$$

where $P(\xi)$ is the Gaussian distribution (6) for ξ. To equip (17) with the initial condition, put in (14) $\gamma = \Gamma$ and get $\tilde{Z}(t, \gamma = \Gamma, \Gamma) = \mathrm{e}^{tp^{-\alpha\Gamma}\xi_\Gamma}$. Thus, we fix the initial condition at the root of the tree by

$$G_\Gamma(x) = \langle\exp\{-\exp[t(1-p^{-1})p^{-\alpha\Gamma}x]\exp[tp^{-\alpha\Gamma}\xi_\Gamma]\}\rangle_{P(\xi_\Gamma)}. \tag{18}$$

The linear approximation in (14) implies the narrow distribution $P(\xi)$ with $\sigma^2 \ll 1$. In this case, the boundary condition (18) reads as

$$G_\Gamma(x) = \exp\{-\exp[t(1-p^{-1})p^{-\alpha\Gamma}x]\}. \tag{19}$$

Equations (17) and (19) set the problem. It is more convenient to revert the direction along a tree by defining $n = \Gamma - m$ $(0 \leqslant n \leqslant \Gamma - \gamma)$. The case $\alpha > 0$ is a 'contracting map' and will be analyzed elsewhere, while here we pay attention to the case $\alpha = 0$ (i.e. for $\langle q_\gamma^{(n)}\rangle = p^{-\gamma}$), which is formally identical to the Derrida–Spohn model of DPDT [14]. It is known from [14] that the solution of (17) and (19) in the continuum approximation is related to the Kolmogorov–Petrovsky–Piscounov (KPP) equation [15] and has a traveling wave.

In terms of the work [14], for long distances on the Cayley tree, the partition function, $G_n(x)$, is a traveling wave of the form $G_n(x) = w(x - fn)$, where the speed f is fixed by the initial condition, i.e. by the 'inverse temperature', β, where $\beta = t(1-p^{-1})$ (for $\alpha = 0$). The speed, f, is the free energy of the system per unit length in the long-length limit $f(t) = \frac{1}{\beta}\ln\left(p\int \mathrm{d}\xi\, P(\xi)\,\mathrm{e}^{\beta\xi}\right)$. For $t < t_{\mathrm{cr}}$ the traveling wave propagates with the speed $f(t)$, while for $t > t_{\mathrm{cr}}$ the speed f is freezed at the critical value $f(t_{\mathrm{cr}})$. The value t_{cr} is determined by the solution of the equation $\frac{\mathrm{d}}{\mathrm{d}t}f(t)\big|_{t=t_{\mathrm{cr}}} = 0$. The overlap between two trajectories starting from the common root point can be (i) 0 (with probability 1) for $t < t_{\mathrm{cr}}$ and (ii) either 0 (with probability $\pi(t) = \mathrm{const}/t$) or 1 (with probability $1 - \pi(t)$) for $t > t_{\mathrm{cr}}$.

Despite that the formal correspondence of our problem for $\alpha = 0$ with DPDT is complete, the interpretation of the behavior observed in [14] in terms of our model is not straightforward and deserves some discussion. The principal difference between the partition function (14) and that of DPDT consists in the following. Our kinetic problem is defined on a p-adic tree, i.e. *only on the boundary* of an ultrametric p-adic the Cayley tree and all kinetic properties of our model deal with probabilities of covering some distance *along a tree's boundary* for a given time interval. In contrast, the DPDT model is defined *in the 'bulk' of the Cayley tree*, i.e. on the whole set of tree's generations, m, and the traveling wave propagates along m, from the root point, $m = 0$, 'downwards'.

The overlap of the trajectories computed in [14] allows one to connect the 'bulk' and the 'boundary' behaviors. Since on small time scales $t < t_{cr}$ (i.e. for 'high temperatures') the overlap of any two trajectories (starting from the root of the tree) is zero, the boundary of the Cayley tree is uniform since there is no clustering. Hence, the escape probability from any basin at the boundary of the Cayley tree is independent of the specific point on this boundary and the kinetics happens as for a nonrandom Parisi matrix. In contrast, in large time scales $t > t_{cr}$ (i.e. for 'low temperatures') the overlap of the trajectories signals the 'probabilistic non-uniformity' of the tree boundary: with the probability $\pi(t)$ the boundary is still uniform, while with the probability $1 - \pi(t)$ the boundary has a 'lacunary' structure such that escape from some boundary basins is locked and the kinetics is confined in these basins for an infinitely long time.

The linear approximation (3) applied to (14) deserves some special care. Actually, due to (3) an unphysical positive tail in the spectral density $\rho(\lambda)$ could appear. Since the survival probability $\overline{W}(t, \Gamma)$ depends on all eigenvalues (see equation (12)), for sufficiently large t these unphysical tails could have a non-negligible effect on the observed behavior. In order to eliminate such an influence, let us note that as σ^2 decreases, the linearized behavior becomes better (see figure 2). So, we fix σ and define the maximal time, $t_{max}(\sigma)$, until which our predictions are valid and the interference of an unphysical tail is negligible. The estimation of $t_{max}(\sigma)$ is very straightforward. Namely, it follows from (12) that the major contribution to $\overline{W}(t, \Gamma)$ at large t comes from the relaxation modes, for which $t\lambda_\gamma^{(n)} \sim 1$. At the same time, one sees from (9) that the unphysical tail corresponding to $\lambda > 0$ is exponentially small for $v_\gamma \gg \sigma\sqrt{u_\gamma}$. For practical purposes, we may use the estimate $v_\gamma \gtrsim 2\sigma\sqrt{u_\gamma}$. Using (8), we get $\sigma \lesssim 0.5$. Thus, for $\sigma \lesssim 0.5$ the influence of an unphysical tail is exponentially suppressed for any t.

In this paper, we have analyzed the simplest statistical properties of randomized Parisi-type kinetic matrices. We also believe that the observed relation between dynamics on ultrametric landscapes described by LCP matrices and directed polymers on disordered trees is very promising for establishing a connection between the elements of p-adic analysis and disordered models on tree-like structures. Our work should be considered a preliminary step in this direction. The straightforward generalization of the obtained results concern an explicit account for the terms of order of $\left|\xi_\gamma^{(n)}\right|^2$ in (3) for the spectral density and for the survival probability.

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